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SUMMARY

Carrying on our research on the deuteron relativistic wave function, we have applied Bethe and Salpeter's relativistic wave equation to the bound system of two particles with spin $\frac{1}{2}$. After some very slight approximations, the solution of the infinite system of coupled integral equations reduces to the explicit structure of the wave matrices corresponding to the S and D states. The percentage of D state obtained is in good agreement with the experimental result.

I. INTRODUCTION

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We have already studied the problem of the determination of a relativistic deuteron wave function [1]. In this paper, we investigate the case of two Dirac nucleons with the Bethe and Salpeter equation formalism[2]. The expansion of the wave function in hyperspherical harmonics has been first introduced in the nucleon-nucleon scattering problem [3]. It was used in our first paper for a spinless deuteron. We employ here the PS (PS) meson theory with an interaction invariant under rotations.

We obtain an infinite coupled system of radial equations for the 16 components of the wave function. In the first section, we develop some calculations in order to obtain the equations corresponding to the $J = 1$ bound neutron-proton state. Because of parity conservation, we can use selection rules to simplify the equations.

In the second section, we analyze the structure of the integral equations. There appear three quantum numbers L, m, n due to the expansion in hyperspherical harmonics, of the three angles θ, φ, β, corresponding to the four-vector $\underset{\sim}{p}$. The first one L is the orbital angular momentum ; the second m, its projection over Oz. The last one n has no direct physical interpretation ; its parity is conserved and it is responsible for the coupling between integral equations. But it is very useful to note that this coupling is extremely weak ; practically the lowest value for n is important and we can reduce the determination of the wave functions to the resolution of one system of two homogeneous coupled integral equations. The splitting of the wave matrix into S and D parts becomes trivial.

In the third section, we use various methods to solve these equations. The eigenvalue of the integral kernel possesses only discrete values and the lower one is proportional to the coupling constant of nuclear forces. The computations performed in the ladder approximation leads to values of $\frac{g^2}{4\pi}$ less than 8 (experimental value $\simeq 15$). The inclusion in the interaction kernel of

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1) M. Gourdin and J. Tran Thanh Van, Nuovo Cimento 14, 1051 (1959)
2) E. E. Salpeter and H. A. Bethe , Phys. Rev. 84, 1232 (1951)
3) M. Gourdin, - Annales de Physique (1959)
- Nuovo Cimento 7, 338 (1958)

fourth-order terms appears as necessary as prohibitive from a practical point of view.

In the last section, we present a simplified method to obtain a relativistic wave function. The relativistic corrections are divided into two types. The kinematical corrections and the properties of symmetry correctly describe the spin structure of the Deuteron which is represented by an entirely known matrix. The relativistic corrections about the dynamics of the system are partially included in a scalar function which is equivalent, in a non relativistic limit, to the Hulthén's wave function. We hope this solution, essentially phenomenological, is reasonable and sufficiently simple to permit calculation. The corresponding percentage of D state in the deuteron is 4 °/o in agreement with experimental results.

II INTEGRAL EQUATIONS for the DEUTERON WAVE FUNCTION

II. 1. System of two bound nucleons with spin $\frac{1}{2}$

The nucleon is represented by a four-component spinor and the two-nucleon wave function $\Phi(\underset{\sim}{p})$ is a 4 x 4 matrix, the 'direct product' in a loose sense of the two spinors for the two particles.

This matrix satisfies the Bethe and Salpeter equation

$$\Phi(\underset{\sim}{p}) = \alpha \frac{(\gamma^{(1)}\underset{\sim}{p}_1 + iM)(\gamma^{(2)}\underset{\sim}{p}_2 + iM)}{(p_1^2 + M^2)(p_2^2 + M^2)} \gamma_5^{(1)} \gamma_5^{(2)} \int W(\underset{\sim}{p},\underset{\sim}{p}') \Phi(\underset{\sim}{p}') \, d\underset{\sim}{p}' \qquad (1)$$

if we consider a PS (PS) coupling.

The four-moments $\underset{\sim}{p}_1$, $\underset{\sim}{p}_2$ are related to the moment of the center of mass $\underset{\sim}{P}$ and the relative moment $\underset{\sim}{p}$ through the equations :

$$\underset{\sim}{P} = \underset{\sim}{p}_1 + \underset{\sim}{p}_2 \qquad 2\,\underset{\sim}{p} = \underset{\sim}{p}_1 - \underset{\sim}{p}_2$$

or

$$\underset{\sim}{p}_1 = \frac{\underset{\sim}{P}}{2} + \underset{\sim}{p} \qquad \underset{\sim}{p}_2 = \frac{\underset{\sim}{P}}{2} - \underset{\sim}{p}$$

The matrices $\gamma_\mu^{(i)}$ relative to the particle (i) act on $\underset{(4)}{\Phi}(p)$ from the left if i = 1 and, after transposition, to the right if i = 2 .

$$\gamma_\mu^{(1)} \Phi(\underset{\sim}{p}) = \gamma_\mu \Phi(\underset{\sim}{p})$$

$$\gamma_\mu^{(2)} \Phi(\underset{\sim}{p}) = \Phi(\underset{\sim}{p}) \gamma_\mu^T$$

γ_μ being an usual Dirac matrix and γ_μ^T , its transpose.

Let us introduce the B matrix :

$$\gamma_\mu^T = B^{-1} \gamma_\mu B$$

and the wave function

$$\underset{I}{\Psi}(\underset{\sim}{p}) = \Phi(\underset{\sim}{p}) B^{-1} ,$$

then we get the simple relations :

$$\gamma_\mu^{(1)} \underset{I}{\Psi} = \gamma_\mu \underset{I}{\Psi}$$

$$\gamma_\mu^{(2)} \underset{I}{\Psi} = \underset{I}{\Psi} \gamma_\mu .$$

The new wave matrix $\underset{I}{\Psi}(\underset{\sim}{p})$ satisfies the integral equation :

$$\underset{I}{\Psi}(\underset{\sim}{p}) = \frac{\alpha}{(p^2 + M^2 + \frac{p^2}{4}) - p_4^2 p^2} \int W(\underset{\sim}{p}, \underset{\sim}{p'}) \left[(\gamma \underset{\sim}{p_1} + iM) \gamma_5 \underset{I}{\Psi}(\underset{\sim}{p'}) \gamma_5 (\gamma \underset{\sim}{p_2} + iM) \right] d\underset{\sim}{p'} \qquad (2)$$

Let us write $\underset{I}{\Psi}(\underset{\sim}{p})$ in a reduced form having four elements which are 2×2 matrices

$$\underset{I}{\Psi}(\underset{\sim}{p}) = \begin{pmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{pmatrix}$$

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4) Louis de Broglie : Théorie générale des particules à Spin (Méthode de fusion)

Let us now express the 2 × 2 matrices in terms of the unit matrix and the Pauli matrices

$$\Psi_{11}(\underset{\sim}{p}) = S_1(\underset{\sim}{p})\, I + \vec{\sigma}\,\vec{V}_1(\underset{\sim}{p})$$

$$\Psi_{22}(\underset{\sim}{p}) = S_2(\underset{\sim}{p})\, I + \vec{\sigma}\,\vec{V}_2(\underset{\sim}{p})$$

$$\Psi_{12}(\underset{\sim}{p}) = \frac{B(\underset{\sim}{p}) + C(\underset{\sim}{p})}{2}\, I + \vec{\sigma}\,\frac{\vec{F}(\underset{\sim}{p}) + \vec{G}(\underset{\sim}{p})}{2}$$

$$\Psi_{21}(\underset{\sim}{p}) = \frac{B(\underset{\sim}{p}) - C(\underset{\sim}{p})}{2}\, I + \vec{\sigma}\,\frac{\vec{F}(\underset{\sim}{p}) - \vec{G}(\underset{\sim}{p})}{2}\ .$$

If we substitute these expressions for the reduced elements of $\Psi(\underset{\sim}{p})$ in equ. (2) we get a system of coupled homogeneous integral equations which has already been written down for the scattering problem [3].

Using the same method as for the problem of two particles without spin [1] we expand the scalar functions $S_1(\underset{\sim}{p})$, $S_2(\underset{\sim}{p})$, $B(\underset{\sim}{p})$, $C(\underset{\sim}{p})$ in hyperspherical harmonics and the vector functions in vectorial hyperspherical harmonics. We assume, as was done before, that the interaction kernel $W(\underset{\sim}{p},\underset{\sim}{p'})$ depends only on the lengths of the four vectors p, p' and on the angle $(\underset{\sim}{H})$ between them.

This assumption allows us to integrate over the angles θ and φ, the hyperspherical harmonics, both scalar and vectorial, being orthonormalized. When performing the integration over the third angle β we are led to introduce a lot of auxiliary functions which are the elements of a 16 × 16 matrix which we call $K(p\ ;\ J,\ n,\ n')$. This matrix will eventually act on the 16 components vector $\Psi(\underset{\sim}{p})$ which was written as a 4 × 4 matrix itself. In the case of nucleons without spin, K was shown to be simply the function $E_0(p\ ;\ l,\ n,\ n')$.
We order the 16 components of $\Psi(\underset{\sim}{p})$ in an arbitrary way ; the non vanishing elements of K indicate between what radial components of $\Psi(\underset{\sim}{p})$ a coupling does exist. We have established in a former work on the nucleon-nucleon scattering [3] the following results :

a) The K matrix is expressed in terms of four 8 x 8 sub-matrices. The two non-diagonal ones are zero, the two diagonal ones refer to a definite value of the parity, the total angular momentum J being a constant of the motion.

b) The sub-matrix of parity $(-1)^J$ is in turn expressed in the same way, in terms of two 4 x 4 sub-matrices lying along the diagonal referring respectively to singlet (J = L, m = 0) and triplet (J = L, m = ± 1) states. m is the azimuthal quantum number.

II. 2. The deuteron case : J = 1 state

For a deuteron in the ground state, J = 1 and parity is + 1. The wave function will, a priori, have 8 components and K matrix will be the sub-matrix 8 × 8 of parity + 1.

Let us write in the following order the radial components A(p ; J L m n) of $\Psi(\underset{\sim}{p})$, omitting the index J equal to 1 and the index L when L is equal to 1

$$\begin{array}{llll} V_1(p\,;\,2\,m\,n)\,; & V_1(p\,;\,0\,m\,n)\,; & V_2(p\,;\,2\,m\,n)\,; & V_2(p\,;\,0\,m\,n) \\ B\,(p\,;\,m\,n) & G\,(p\,;\,m\,n)\,; & C\,(p\,;\,m\,n) & F\,(p\,;\,m\,n). \end{array} \qquad (3)$$

On the other hand, the matrix K(p ; n n') is an integral over the angle β of another matrix R(p, β) :

$$K_{\lambda\mu}(p\,;n,\,n') = \int_0^{\pi} \frac{\mathcal{C}_{\ell}^{n}(\beta)\;\; R_{\lambda\mu}(p,\,\beta)\;\; \mathcal{C}_{\ell'}^{n'}(\beta)}{(p^2 + M^2 + \frac{p^2}{4})^2 - p^2\,p^2\cos^2\beta}\,\sin^2\beta\,d\beta \qquad (4)$$

The indices ℓ, ℓ' are related respectively to the components λ, μ. With the above choice (3) for numbering the radial components $A_{\lambda}(p\,;\,1,\,m,\,n)$ we have

$$\begin{array}{l} \ell = 0 \quad \text{for} \quad \lambda = 2,\,4 \\ \ell = 1 \quad \text{for} \quad \lambda = 5,\,6,\,7,\,8 \\ \ell = 2 \quad \text{for} \quad \lambda = 1,\,3\,. \end{array} \qquad (5)$$

The matrix R is given in the table I .

Table I : R Matrix

$$R_{11} = -\frac{p^2 \sin^2\beta}{3} \qquad R_{12} = 2\sqrt{2}\, R_{11} \qquad R_{13} = -\left[p^2 \cos^2\beta + (2M - \frac{B}{2})^2\right] \qquad R_{14} = 0$$

$$R_{15} = \sqrt{\frac{2}{3}}\, p \sin\beta\, (2M - \frac{B}{2}) \qquad R_{16} = \frac{1}{\sqrt{2}} R_{15} \qquad R_{17} = i\sqrt{\frac{2}{3}}\, p^2 \sin\beta \cos\beta \qquad R_{18} = \frac{1}{\sqrt{2}} R_{17}$$

$$R_{21} = R_{12} \qquad R_{22} = -R_{11} \qquad R_{23} = 0 \qquad R_{24} = R_{13}$$

$$R_{25} = R_{16} \qquad R_{26} = -R_{15} \qquad R_{27} = R_{18} \qquad R_{28} = -R_{17}$$

$$R_{31} = -\left[p^2 \cos^2\beta + \frac{B^2}{4}\right] \qquad R_{32} = 0 \qquad R_{33} = R_{11} \qquad R_{34} = R_{12}$$

$$R_{35} = -\sqrt{\frac{2}{3}}\, p \sin\beta\, \frac{B}{2} \qquad R_{36} = -\frac{1}{\sqrt{2}} R_{35} \qquad R_{37} = R_{28} \qquad R_{38} = -R_{27}$$

$$R_{41} = 0 \qquad R_{42} = R_{31} \qquad R_{43} = R_{21} \qquad R_{44} = +R_{22}$$

$$R_{45} = -R_{35} \qquad R_{46} = R_{35} \qquad R_{47} = R_{38} \qquad R_{48} = R_{17}$$

$$R_{51} = -2R_{35} \qquad R_{52} = 2R_{36} \qquad R_{53} = -2R_{15} \qquad R_{54} = -2R_{16}$$

$$R_{55} = p^2 - \gamma^2 \qquad R_{56} = 0 \qquad R_{57} = -2\, i\, p\, M \cos\beta \qquad R_{58} = 0$$

$$R_{61} = R_{52} \qquad R_{62} = -R_{51} \qquad R_{63} = -R_{54} \qquad R_{64} = R_{53}$$

$$R_{65} = 0 \qquad R_{66} = -R_{55} \qquad R_{67} = 0 \qquad R_{68} = -R_{57}$$

$$R_{71} = -2R_{17} \qquad R_{72} = -2R_{18} \qquad R_{73} = 2R_{17} \qquad R_{74} = 2R_{18}$$

$$R_{75} = -R_{57} \qquad R_{76} = 0 \qquad R_{77} = -p^2 \cos 2\beta + \gamma^2 \qquad R_{78} = 0$$

$$R_{81} = -2R_{18} \qquad R_{82} = 2R_{28} \qquad R_{83} = -2R_{18} \qquad R_{84} = 2R_{17}$$

$$R_{85} = 0 \qquad R_{86} = R_{57} \qquad R_{87} = 0 \qquad R_{88} = -R_{77}$$

The selection rules due to the parity of the Gegenbauer polynomials show that certain elements of K will be zero after integration over β. Thus, the components A_1, A_2, A_3, A_4, A_5, A_6 will be real while the components A_7, A_8 will be imaginary ; we have $n + n' = \ell + \ell' + 2q$ for the former and $n + n' = \ell + \ell' + 2q + 1$ for the latter.

Then, we obtain for the radial components $A_\lambda (p ; m, n)$ an infinite system of integral equations coupled by the indices n and λ .

$$A_\lambda (p ; m\ n) = \alpha \sum_{n'} \sum_{\mu} K_{\lambda\mu}(p ; n\ n') \int_0^{\infty} \Delta_{n'}(p,p')\ A_\mu(p' ; m\ n')\ p'^3\ dp' \qquad (6)$$

III APPROXIMATE RESOLUTION of THE SYSTEM OF INTEGRAL EQUATIONS

Using the same approximation methods as in the preceding article [1] we propose to find a solution to the system of integral equations (6). We saw that these equations were very weakly coupled with respect to the index n and that the wave functions fell off very fast as n increased. The function $W(\underset{\sim}{p},\underset{\sim}{p}')$ which is the scalar part of the interaction without the Dirac matrices corresponds to the same function $W(\underset{\sim}{p},\underset{\sim}{p}')$ introduced for the simplified scalar problem. Thus it turns out that the functions $\Delta_n (p,p')$ are the same in the two cases. Actually we have employed the ladder approximation but the arguments for the convergence of series with the index n remain valid for physically reasonable interactions. The falling off of $\Delta_n (p,p')$ with n is very fast, we shall decouple the equations and conserve for a given component only the term of the smallest order in n.

III.1. Terms n = 0 and n' = 0

The relation connecting ℓ, ℓ', n, n' and q shows that, in this case, only two components intervene : $V_1(p ; 0\ 0\ 0)$ and $V_2(p ; 0\ 0\ 0)$ which we call $V_1^-(p)$ and $V_2^-(p)$. The system of integral equations (6) is then reduced to the two coupled equations :

$$V_1^-(p) = \alpha\left[K_{22}(p)\int_0^\infty \Delta_0(p,p')\, V_1^-(p')\, p'^3 dp' + K_{24}(p)\int_0^\infty \Delta_0(p,p')\, V_2^-(p')\, p'^3 dp'\right]$$

$$V_2^-(p) = \alpha\left[K_{42}(p)\int_0^\infty \Delta_0(p,p')\, V_1^-(p')\, p'^3 dp' + K_{44}(p)\int_0^\infty \Delta_0(p,p')\, V_2^-(p')\, p'^3 dp'\right] \tag{7}$$

Henceforth, by convention, $K_{\lambda\mu}(p)$ is the function $K_{\lambda\mu}(p\,;\,n,\,n')$ corresponding to minimum values of n and n' for which it is not zero. Their calculation becomes that of the functions $E_0(p\,;\,\ell\ 0\ 0)$.

$$K_{22}(p) = K_{44}(p) = \frac{p^2}{4}\left(1 - \frac{h(p)}{3}\right) E_0$$

$$K_{24}(p) = -\left[\frac{p^2}{4}(1 + h(p)) + 4M^2\right] E_0(p)$$

$$K_{42}(p) = -\left[\frac{p^2}{4}(1 + h(p)) + \left(\frac{\gamma^2}{4M}\right)^2\right] E_0(p)\ .$$

Recall

$$E_0(p) = \frac{2}{(p^2+\gamma^2)\left\{(p^2+\gamma^2)^2 + 4p^2M^2\right\}^{1/2}}$$

$$h(p) = \frac{(p^2+\gamma^2) - \left\{(p^2+\gamma^2)^2 + 4p^2M^2\right\}^{1/2}}{(p^2+\gamma^2) + \left\{(p^2+\gamma^2)^2 + 4p^2M^2\right\}^{1/2}}$$

and the binding energy of deuteron -B appears through the parameter γ thus defined

$$\gamma^2 = MB\ .$$

To facilitate calculation we define two auxiliary functions $I_1(p)$ and $I_2(p)$

$$I_{1,2}(p) = \int_0^\infty \Delta_0(p,p')\ V_{1,2}^-(p')\, p'^3\, dp' \tag{8}$$

permitting us to write the system (7) as

$$V_1^- (p) = \alpha \left[K_{22}(p)\, I_1(p) + K_{24}(p)\, I_2(p) \right]$$
$$V_2^- (p) = \alpha \left[K_{42}(p)\, I_1(p) + K_{22}(p)\, I_2(p) \right] \qquad (9)$$

III. 2. Terms n = 1, n' = 0

At this order of the approximation, two new components of the wave matrix B(p ; 1 m 1) and G(p ; 1 m 1) appear. We call them B (p) and G (p) and express them with the help of I_1 and I_2 .

$$B (p) = \alpha \left[K_{52}(p)\, I_1(p) + K_{54}(p)\, I_2(p) \right]$$
$$G (p) = \alpha \left[K_{62}(p)\, I_1(p) + K_{64}(p)\, I_2(p) \right] \qquad (10)$$

The four new functions $K_{\lambda\mu}(p)$ are then given in terms of $E_1(p)$ by

$$K_{52}(p) = \left(\frac{\gamma}{2M}\right)^2 2\, p\, M\, E_1(p) \qquad K_{54}(p) = -\, 2\, p\, M\, E_1$$

$$K_{62}(p) = -\left(\frac{\gamma}{2M}\right)^2 2\sqrt{2}\, p\, M\, E_1(p) \qquad K_{64}(p) = -\, 2\sqrt{2}\, p\, M\, E_1(p)$$

where

$$E_1(p) = \left(1 - \frac{h(p)}{3}\right) E_0(p)$$

We can invert the equations (9)

$$I_1 = \frac{1}{\alpha W}\left[K_{22}\, V_1^- - K_{24}\, V_2^- \right]$$

$$I_2 = \frac{1}{\alpha W}\left[-\, K_{42}\, V_1^- + K_{44}\, V_2^- \right]$$

where we put

$$W (p) = K_{22}\, K_{44} - K_{24}\, K_{42}$$

and see that B (p) and G (p) are, at this point in the calculation, simple linear combination of V_1^- (p) and V_2^- (p), the coefficients being expressed in terms of the functions $K_{\lambda\mu}(p)$.

Finally one should notice that the binding energy of the Deuteron being very small (B = 2,23 MeV), the number $\left(\frac{\gamma}{2M}\right)^2$ is much less than unity ($\simeq \frac{1}{2\,000}$) ; $K_{52}(p)$ and K_{62} (p) are thus negligible before $K_{54}(p)$ and $K_{64}(p)$. Doing this, we find a simple relation between B and G

$$G\,(p) = 2\sqrt{B}\,(p) \tag{11}$$

which is certainly valid to a very good approximation.

III. 3. Terms n = 0, n' = 1

The equations (7) are now modified by the addition of a coupling term between V_1^- (p), V_2^- (p) and B (p), G (p). We thus calculate a first correction to the solutions of the equations (7).

If we introduce the auxiliary functions $I_3(p)$ and $I_4(p)$

$$I_3(p) = \int_0^\infty \Delta_1(p,p')\, B\,(p')\, p'^3\, dp'$$

$$I_4(p) = \int_0^\infty \Delta_1(p,p')\, G\,(p')\, p'^3\, dp' ,$$

the system (9) becomes

$$\begin{aligned} V_1^-\,(p) &= \alpha\left[K_{22}\, I_1 + K_{24}\, I_2 + K_{25}\, I_3 + K_{26}\, I_4\right] \\ V_2^-\,(p) &= \alpha\left[K_{42}\, I_1 + K_{44}\, I_2 + K_{45}\, I_3 + K_{46}\, I_4\right] \end{aligned} \tag{12}$$

with

$$K_{25}(p) = p\, M\, E_1 \qquad K_{26}(p) = -\sqrt{2}\, K_{25}(p)$$

$$K_{45}(p) = -\left(\frac{\gamma}{2M}\right)^2 K_{25}(p) \qquad K_{46}(p) = \left(\frac{\gamma}{2M}\right)^2 K_{26}(p)$$

The problem is then to solve the coupled system (10) and (12). By eliminating B (p) and G (p) we obtain two coupled integral equations for $V_1^-(p)$ and $V_2^-(p)$ alone. The difference between this system and (8) will enable us to test our development in series with respect to n.

But, it is legitimate to neglect $\left(\frac{\gamma}{2M}\right)^2$ before unity. Relation (11) is then valid and the system (12) is equivalent to

$$\begin{aligned} V_1^-(p) &= \alpha\left[K_{22}\, I_1 \;+\; K_{24}\, I_2 \;-\; K_{25}\, I_3\right] \\ V_2^-(p) &= \alpha\left[K_{42}\, I_1 \;+\; K_{44}\, I_2\right]. \end{aligned} \tag{13}$$

The second equation (9) is not modified, so that, taking (8),(9),(11) into account, we get only one substitution.

$$K_{24}(p)\ \Delta_0(p,p') \Longrightarrow K_{24}(p)\ \Delta_0(p,p') - K_{25}(p) \int_0^\infty \Delta_1(p,p'')\ K_{54}(p'')\ \Delta_0(p'',p')\ p''^3\ dp''$$

which can be easily calculated in the plane (p, p') since all the elements are known.

III. 4. Terms n = 2, n' = 0 (real)

In this case, the orbital angular momentum components of order 2 appear. They are $V_1(p\ ;\ 2\ m\ 2)$ and $V_2(p\ ;\ 2\ m\ 2)$, which we call $V_1^+(p)$ and $V_2^+(p)$ respectively. They are easily known once $V_1^-(p)$ and $V_2^-(p)$ have been calculated.

$$\begin{aligned} V_1^+(p) &= \alpha\, K_{12}(p)\, I_1(p) \\ V_2^+(p) &= \alpha\, K_{34}(p)\, I_2(p)\ . \end{aligned} \tag{14}$$

The two functions $K_{12}(p)$ and $K_{34}(p)$ are equal

$$K_{12}(p) = K_{34}(p) = -\frac{\sqrt{5}}{3}\ p^2 \left[1 - \frac{h(p)}{2} + \frac{h^2(p)}{10}\right] E_0(p)$$

III. 5. Terms n = 2 n' = 0 (imaginary)

The two imaginary functions that we introduce here $C(p\ ;\ 1\ m\ 2)$ and $F(p\ ;\ 1\ m\ 2)$ are to be multiplied by the function $\mathcal{C}^2_1(\beta)$. It is easy to see that in configuration space, the same thing happens. At the limit of equal times, $\beta = \frac{\pi}{2}$ and $\mathcal{C}^2_1(\frac{\pi}{2}) = 0$. For that reason, we do not worry about these imaginary functions.

III. 6. Conclusion

Taking into account in each case only the lowest terms in n and n', we are led to the following programme

a) Solve the system (7) to obtain $V_1^-(p)$, $V_2^-(p)$, I_1 and I_2 ,

b) Calculate B (p) and G (p) by equations (10) ,

c) Calculate $V_1^+(p)$ and $V_2^+(p)$ by equations (14) .

IV SEPARATION into STATES $^3S_1$ and $^3D_1$

The wave matrix $\Psi(\underset{\sim}{p})$ in our particular case has therefore the form :

$$\Psi(\underset{\sim}{p}) = \begin{vmatrix} \vec{\sigma} \cdot \vec{V}_1(\underset{\sim}{p}) & \frac{B(\underset{\sim}{p})}{2} + \frac{\vec{\sigma}\vec{G}(\underset{\sim}{p})}{2} \\ \frac{B(\underset{\sim}{p})}{2} - \frac{\vec{\sigma}\vec{G}(\underset{\sim}{p})}{2} & \vec{\sigma} \cdot \vec{V}_2(\underset{\sim}{p}) \end{vmatrix}$$

In the state of total angular momentum J = 1, it becomes :

$$\vec{V}_{1,2}(\underset{\sim}{p}) = V^-_{1,2}(p)\ \vec{\mathcal{Y}}^{m\ 0}_{1\ 0\ 1}(\theta, \varphi, \beta) + V^+_{1,2}(p)\ \vec{\mathcal{Y}}^{m\ 0}_{1\ 2\ 1}(\theta, \varphi, \beta)$$

$$B(\underset{\sim}{p}) = B(p)\ \mathcal{C}^1_1(\beta)\ \mathcal{Y}^m_1(\theta, \varphi)$$

$$\vec{G}(\underset{\sim}{p}) = G(p) \cdot \vec{\mathcal{Y}}^{m\ 0}_{1\ 1\ 1}(\theta, \varphi, \beta)$$

The vectorial hyperspherical harmonics $\vec{\mathcal{Y}}^{\,m\,n}_{J\,L\,1}(\theta, \varphi, \beta)$ may be obtained from scalar hyperspherical harmonics by the relations :

$$\vec{\mathcal{Y}}^{\,m\,\vec{n}}_{J\,J\,1} = \frac{1}{\sqrt{J(J+1)}}\ \vec{L}\,\mathcal{Y}^{\,m\,n}_{J}$$

$$\vec{\mathcal{Y}}^{\,m,n}_{J,\,J+1,\,1} = \frac{1}{\sqrt{(J+1)(2J+1)}}\left[(J+1)\,\vec{e}_p + i\,\vec{e}_p \times \vec{L}\right]\mathcal{Y}^{m,n}_{J}$$

$$\vec{\mathcal{Y}}^{\,m,n}_{J,\,J-1,\,1} = \frac{1}{\sqrt{(J+1)(2J+1)}}\left[J\,\vec{e}_p - i\,\vec{e}_p \times \vec{L}\right]\mathcal{Y}^{m,n}_{J}$$

We easily deduce the two relations

$$\mathcal{Y}^{\,m\,n}_{J} = (\vec{\sigma}\,\vec{e}_p)\cdot\vec{\sigma}\left[\sqrt{\frac{J+1}{2J+1}}\ \vec{\mathcal{Y}}^{\,m,n}_{J,\,J+1,\,1} + \sqrt{\frac{J}{2J+1}}\ \vec{\mathcal{Y}}^{\,m,n}_{J,\,J-1,\,1}\right]$$

$$\vec{\sigma}\ \vec{\mathcal{Y}}^{\,m\,n}_{J\,J\,1} = (\vec{\sigma}\,\vec{e}_p)\cdot\vec{\sigma}\left[\sqrt{\frac{J}{2J+1}}\ \vec{\mathcal{Y}}^{\,m\,n}_{J,\,J+1,\,1} - \sqrt{\frac{J+1}{2J+1}}\ \vec{\mathcal{Y}}^{\,m\,n}_{J,\,J-1,\,1}\right]$$

using the property $\vec{e}_p\,\vec{L} = 0$, $\vec{e}_p$ being the unitary vector in the direction $\vec{p}$.

Applying these equations for J = 1, the wave matrix becomes simply :

$$\underset{\sim}{\psi}(p) = M_S(p, \beta)\ \vec{\sigma}\ \vec{\mathcal{Y}}^{\,m}_{1\,0\,1}(\theta,\varphi) + M_D(p, \beta)\ \vec{\sigma}\ \vec{\mathcal{Y}}^{\,m}_{1\,2\,1}(\theta,\varphi)\ .$$

Here $\vec{\mathcal{Y}}^{\,m}_{J\,L\,1}$ are ordinary vector orthonormal spherical harmonics. The wave matrices for the S and D states are given by

$$M_S = \sqrt{\frac{2}{\pi}} \begin{vmatrix} V_1^-(p) & \frac{1}{3}\left[B(p) - \sqrt{2}\ G(p)\right] \sin\beta\ \vec{\sigma}\ \vec{e}_p \\ \frac{1}{3}\left[B(p) + \sqrt{2}\ G(p)\right] \sin\beta\ \vec{\sigma}\ \vec{e}_p & V_2^-(p) \end{vmatrix} \tag{16}$$

$$M_D = \frac{2}{\sqrt{\pi}} \begin{vmatrix} \frac{2}{\sqrt{5}} \sin^2\beta\ V_1^+(p) & \frac{1}{3}\left[B(p) + \frac{1}{\sqrt{2}}\ G(p)\right] \sin\beta\ \vec{\sigma}\ \vec{e}_p \\ \frac{1}{3}\left[B(p) - \frac{1}{\sqrt{2}}\ G(p)\right] \sin\beta\ \vec{\sigma}\ \vec{e}_p & \frac{2}{\sqrt{5}} \sin^2\beta\ V_2^+(p) \end{vmatrix}$$

It is then easy to calculate the proportion of S and D states by the integrals

$$p_S = \frac{\text{Trace} \int |M_S|^2\ p^3\ dp\ \sin^2\beta\ d\beta}{\text{Trace} \int |\Psi(\underset{\sim}{p})|^2\ \underset{\sim}{dp}}$$

$$p_D = \frac{\text{Trace} \int |M_D|^2\ p^3\ dp\ \sin^2\beta\ d\beta}{\text{Trace} \int |\Psi(\underset{\sim}{p})|^2\ \underset{\sim}{dp}} \ .$$

The integration over the angle β are immediate. After calculating the traces we obtain

$$p_S = \frac{N_S}{N_S + N_D} \qquad \text{and} \qquad p_D = \frac{N_D}{N_S + N_D}$$

with

$$N_S = \int_0^\infty \left[|V_1^-(p)|^2 + |V_2^-(p)|^2 + \frac{1}{6} |B(p)|^2 + \frac{1}{3} |G(p)|^2 \right] p^3\ dp$$

$$N_D = \int_0^\infty \left[|V_1^+(p)|^2 + |V_2^+(p)|^2 + \frac{1}{3} |B(p)|^2 + \frac{1}{6} |G(p)|^2 \right] p^3\ dp$$

V NUMERICAL RESOLUTION of INTEGRAL EQUATIONS in the LADDER APPROXIMATION

As we saw before, the possibility to decouple the equations with respect to the index n focuses all the difficulties on the solving of the system of coupled integral equation (7). After fixing the binding energy of deuteron at its experimental value, we intend to determine the coupling constant and the wave functions $V_1^-(p)$ and $V_2^-(p)$ from which we can express all the others by quadratures.

V. 1. Approximate resolution :

We substitute for the exact kernel $\Delta_n(p,p')$ an approximate and separable one :

$$\Delta_n(p,p') \Longrightarrow \frac{\Delta_n(p,q)\,\Delta_n(q,p')}{\Delta_n(q,q)}$$

choosing for q the average momentum of nucleons in the deuteron ($q \simeq \frac{1}{4}\mu$). We can then solve the new system exactly and we find $\alpha = 0{,}13$ i.e. $\frac{g^2}{4\pi}$, the coupling constant of nuclear forces, is equal to 16.

V. 2. Kellog's method of iteration

We have made a numerical calculation on IBM 650 to solve the equations by successive iterations. We thus define a series of values for α at each stage of the iteration. The result is

$$\alpha \simeq 0{,}046 \qquad \text{i.e.} \quad \frac{g^2}{4\pi} \simeq 6 \ ,$$

and the proportion of D state is

$$p_D \simeq 1\ \% \ .$$

We have verified that the develcpment with respect to n is excellent as well. The solving of the system (13) instead of (7) leads us to

$$\alpha \simeq 0{,}048 \qquad \text{i.e.} \qquad \frac{g^2}{4\pi} \simeq 6{,}5 \ .$$

We then obtain a clear disagreement with the experimental value $\frac{g^2}{4\pi} \simeq 15$ deduced for example from meson-nucleon scattering.

The most reasonable explanation for this seems to be the inability of the ladder approximation to describe this phenomenon. It would be necessary in a meson theory to include fourth order terms. Unfortunately this calculation is inextricable and we shall not do it although the formalism remains valid.

VI APPROXIMATE WAVE FUNCTIONS

The calculations with a kernel including the fourth order term being inextricable, we have modified our method in order to obtain a deuteron wave matrix available in concrete problems [5].

If we consider Bethe and Salpeter's equation :

$$\Phi(\underset{\sim}{p}) = \frac{\alpha}{(\gamma^{(1)}p_1 - iM)(\gamma^{(2)}p_2 - iM)} \gamma_5^{(1)} \gamma_5^{(2)} \int W(\underset{\sim}{p},\underset{\sim}{p'}) \Phi(\underset{\sim}{p'}) \, d\underset{\sim}{p'} \ ,$$

we can see that there are 3 types of relativistic corrections :

- the kinematical corrections take the relativistic energy and the spins of the nucleons into account, they are included in the expressions $(\gamma^{(1)}p_1 - iM)$ and $(\gamma^{(2)}p_2 - iM)$;

- the properties of symmetry are correctly represented by the matrices $\gamma_5^{(i)}$ which show that we have used a pseudo-scalar meson-nucleon interaction ;

- the dynamical corrections are included in the expression of the kernel $W(\underset{\sim}{p},\underset{\sim}{p'})$.

In our previous calculation , we have precisely taken the first two

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5) We are indebted to Professor M. Lévy for this suggestion.

kinds of correction into account. On the other hand, we have only partially considered the relativistic effects in the dynamical corrections, assuming the hypothesis that the interaction depends only upon the square of the space-time distance between the two particles. In particular, we have used the ladder approximation.

In separating the two types of corrections, we have obtained :

a) a correct structure of the wave matrix - i.e. the distribution between its different components - due only to the spin of the nucleons and to the pseudo-scalar character of the interaction,

b) the wave functions $V_1^-(p)$, $V_2^-(p)$, which are approximative, but which take some relativistic effects into account.

In order to simplify the integral equations, we keep only the leading terms, and this enables us to obtain a simpler structure of the wave matrix. Considering that

$$\left(\frac{\gamma}{2M}\right)^2 \ll 1$$

and $$V_1^-(p)\ V_2^-(p) = -B^2(p)$$

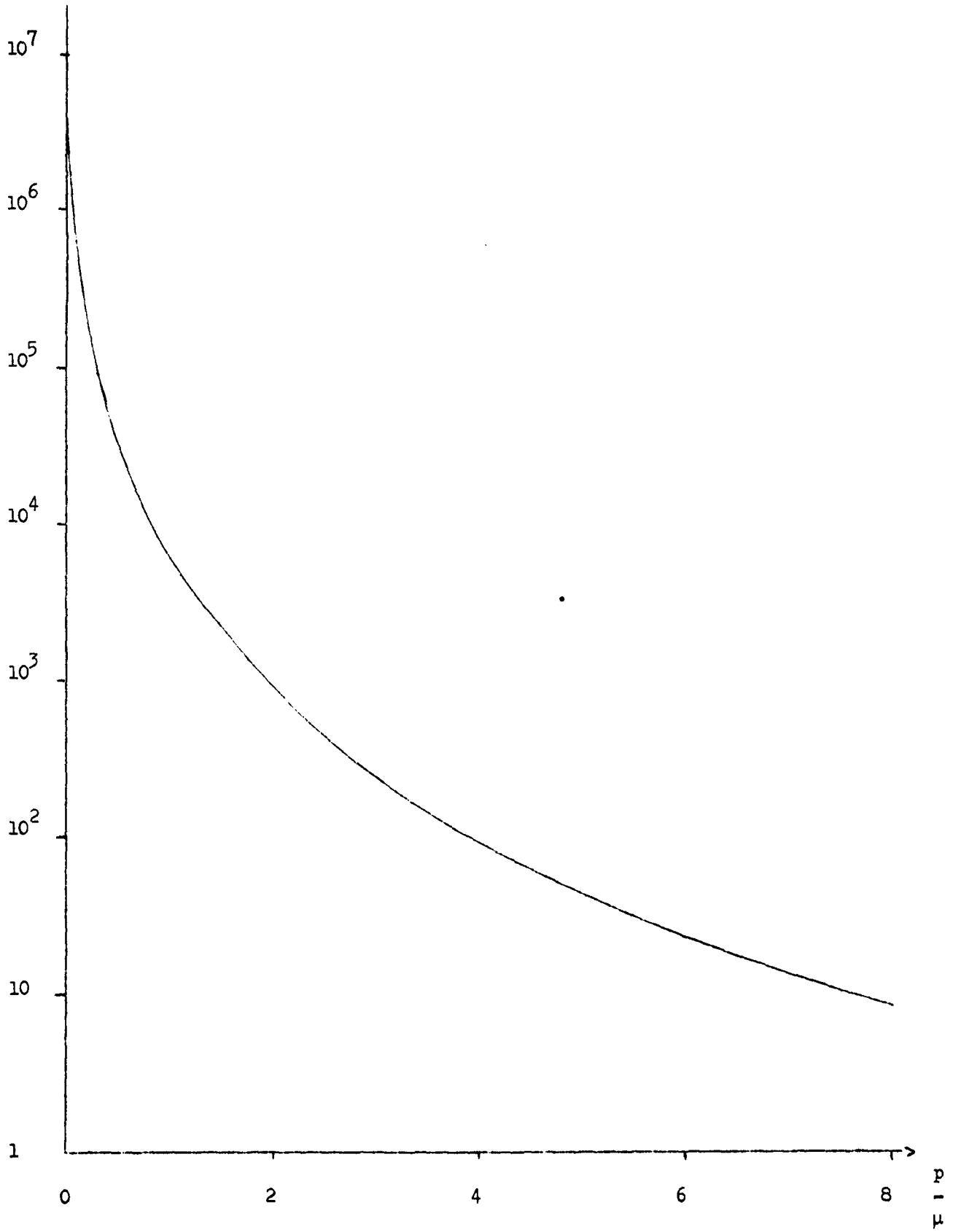
with a very good approximation, we can obtain the wave matrices M_S and M_D

$$M_S = \sqrt{\frac{2}{\pi}} \begin{vmatrix} 1 & -\frac{1}{3}\left(\frac{p}{2M}\right)\left(1-\frac{h}{3}\right)\sin\beta\,(\vec{\sigma}\,\vec{e}_p) \\ \frac{p}{2M}\left(1-\frac{h}{3}\right)\sin\beta\,\vec{\sigma}\,\vec{e}_p & -\left(1-\frac{h}{3}\right)^2\left(\frac{p}{2M}\right)^2 \end{vmatrix} V_1^-(p)$$

$$M_D = \frac{2}{\sqrt{\pi}} \begin{vmatrix} -\frac{8}{3}\,\frac{1-\frac{h}{2}}{1+h}\left(1-\frac{h}{3}\right)^2\left(\frac{p}{2M}\right)^2\sin^2\beta & \frac{2}{3}\left(\frac{p}{M}\right)\left(1-\frac{h}{3}\right)\sin\beta\,\vec{\sigma}\,\vec{e}_p \\ 0 & \frac{2}{3}\left(1-\frac{h}{2}\right)\left(\frac{p}{2M}\right)^2\sin^2\beta \end{vmatrix} V_1^-(p)$$

(18)

Fig. I : V_1^- (p) in arbitrary units



The only remaining unknown scalar function $V_1^-(p)$ corresponds, in the non relativistic approximation, to the wave function for a deuteron without spin. We represent it graphically in fig. I.

The calculation of the immixture of D state gives 4 °/o, which is in satisfactory agreement with experiments. We think that this very phenomenological determination of the deuteron wave function is better than the preceding one. It amounts to writing the wave matrix with the help of a scalar function resembling that of Hulthén, and known matrices describing a structure bound to the existence of nucleon spin. In other words, the matrices M_S and M_D express the coupling between the states $^3S_1$ and $^3D_1$ of Deuteron and include relativistic effects inside the Deuteron which are included in $V_1^-(p)$, a function of the same type as $A_0(p)$, the solution corresponding to the problem without spins.

VII CONCLUSION

We have developed a formalism to determine the relativistic deuteron wave function and the coupling constant of nuclear forces from the binding energy of the deuteron which has been experimentally determined.

We have used the interaction given by the meson theory about nuclear forces in the ladder approximation of second order. The theoretical results make us foresee a coupling constant twice too small and this does not allow us to take too seriously the results obtained about the wave function.

We have tried to consider the mass of meson π as a phenomenological parameter. The most favourable results, corresponding to $\mu = 0{,}1$ ($g^2/4\pi \simeq 7$ and $p_D \simeq 1{,}5$ °/o) show that the coupling constant as well as the proportion of D state are parameters which vary extremely slowly with μ. Besides, this has been verified in the diffusion problem.

We have also thought of adjoining an arbitrary phenomenological term respecting the conditions of invariance to the mesonic term of second order ;

an attempt has been made but was not very conclusive ; perhaps, there are, after all, some attempts to be made in this direction, but the justification and the theoretical origin of such terms seem to be extremely problematical and finally doomed to failure.

With extremely weak approximations, it is possible to give to the wave matrix a structure entirely known from one scalar function only. This method leads us to a very simple form and will permit numerical computations. Particularly, the most immediate and interesting application is the calculation of the elastic electron deuteron scattering cross section in the impulse approximation. The problem becomes very clear to understand. We know how the interaction acts on the neutron and on the proton and all the trace calculations can be performed before integration. The modifications due to the spin of the nucleons will appear naturally and the relativistic corrections will be rapidly evaluated.